QUANTUM BREAKING TIME SCALING IN THE SUPERDIFFUSIVE DYNAMICS

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We show that the breaking time of quantum-classical correspondence depends on the type of kinetics and the dominant origin of stickiness. For sticky dynamics of quantum kicked rotor, when the hierarchical set of islands corresponds to the accelerator mode, we demonstrate by simulation that the breaking time scales as $\tau_\hbar \sim (1/\hbar)^{1/\mu}$ with the transport exponent $\mu > 1$ that corresponds to superdiffusive dynamics [17]. We discuss also other possibilities for the breaking time scaling and transition to the logarithmic one $\tau_\hbar \sim \ln(1/\hbar)$ with respect to $\hbar$.
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1. Classical chaotic dynamics can be characterized by a Lyapunov exponent Λ and infinitely divisible filamentation of the phase flux. Quantized procedure stops the filamentation due to the uncertainty principle and, as a result, breaks the applicability of semiclassical approximaiton. The corresponding breaking time was found in [1]

$$\tau_\hbar = (1/\Lambda)\ln(I_0/\hbar), \tag{1}$$

where I_0 is a characteristic action, indicating a fast (exponential) growth of quantum corrections to the classical dynamics due to chaos. The origin of this time was explained in detail in [2] and gained a wide discussion [3]. The logarithmic scaling in $\hbar$ for $\tau_\hbar$ corresponds to a fairly good and uniform chaotic mixing (see also a final comment in [4]).

Typical Hamiltonian chaotic dynamics is not ergodic due to the presence of infinite islands set in phase space [5], and Lyapunov exponent is not uniform due to cantori and possible hierarchical structures of islands [5] and their stickiness [6,7]. This complicates a process of diffusion transforming the transport from the Gaussian type to the anomalous (fractional) one [8,9]. Particularly, sticky properties of the island boundaries should impose algebraic laws of the survival probability

$$P(t) \sim 1/t^\gamma \tag{2}$$

of a particle to escape after time t from a domain near the islands [8,7]. The immediate consequence from the scaling property (2) is its breaking at some critical time τ^* for the case of quantum chaos since a hierarchical dynamical chain has no limit $t \to \infty$ and should be abrupted when an island in the chain reaches area $S^* = \hbar$ and the quantum effects become important. This comment, starting from [10], was discussed in detail in [4], where a power law for the Planck constant scaling was suggested

$$\tau_\hbar^* \sim 1/\hbar^\delta \tag{3}$$

for the breaking time of classical considertion applicability with a value of δ depending on the type of the classical algebraic escape time distribution. There were no quantum simulation in these works and no definite values of δ. Discussion of the algebraic law (3) started in [4,10], was continued in recent publications [11–16] on the basis of simulations of quantum maps with an explicit evaluation of (2).

The questions arise: What is the actual scaling of breaking time of quantum-classical correspondence with respect to the Planck constant, logarithmic or algebraic? How universal are values of γ or δ? Using some results of [17,18] on the strong delocalization effects, and simulation for the quantum kicked rotor (QKR) we will show here that the result (1), being valid for the cases of normal (Gaussian type) diffusion, appears to be an algebraic of the type (2) when diffusion becomes anomalous, i.e. superdiffusion, with the second moment of the truncated distribution function as

$$\langle p^2 \rangle \sim t^\mu \tag{4}$$

and transport exponent $\mu > 1$. We were able to obtain $\tau_\hbar^*$ from the simulation as a crossover time of the survival probability that changes the exponent of its algebraic behavior, and compare values of δ, predicted by the theory, with the one obtained from the simulation.

2. We consider QKR that corresponds to the standard map in the classical limit

$$p' = p + K\sin q\ , \qquad q' = q + p' \tag{5}$$

defined on the cylinder $p \in (-\infty, \infty)$, $q \in (-\pi, \pi)$ with a control parameter K and the Lyapunov exponent $\Lambda \sim 1/\ln K$ for $K \gg 1$ and for almost all domain excluding areas where $K|\cos q| < 1$. The marginally stable points are defined by the conditions $K_m = 2\pi m$, $p = 2\pi n$, $q = \pm\pi/2$ with integers (m, n). For

$$0 < K - K_m < \Delta K_m \tag{6}$$

a new set of islands appears [6,19,20], called tangle islands in [20], as a result of bifurcation. Dynamics inside the islands is known as the accelerator mode [21,22] and we will call them accelerator mode islands (AMI). Changing of K within the interval (6) influences strongly the topological structure of AMI and consequently the values of the transport exponent $\mu = \mu(K)$ [23,9], since the stickiness of trajectories to the islands boundaries can be different for different island topologies.

It was established in [9] that for a special ("magic") value of $K \equiv K^* = 6.908745\ldots$ the stickiness can be especially what makes this case to be convenient to study properties of the anomalous transport. For the K^* it appears a hierarchical set of islands-around-islands with the islands sequence $3 - 8 - 8 - 8 - \ldots$. The islands chain satisfies the renormalization conditions

$$\begin{aligned} S^{(n+1)} = \lambda_S S^{(n)} \ , \qquad T^{(n+1)} &= \lambda_T T^{(n)} \ , \\ N^{(n+1)} &= \lambda_N N^{(n)} \ , \end{aligned} \tag{7}$$

where n is a number in the hierarchy sequence, $S^{(n)}$ is an island area, $T^{(n)}$ is a period of the last invariant curve of the corresponding island, $N^{(n)}$ is a number of islands in the chain of the n-th hierarchy level, and $\lambda_S < 1$, $\lambda_T > 1$, $\lambda_N > 1$ are some scaling parameters. The renormalization transform (7) can be also extended for Lyapunov exponents Λ in a sticky area of the island's boundary of the n-th level of the island's hiearchy [23]:

$$\Lambda^{(n+1)} = \lambda_L \Lambda^{(n)} = \lambda_L^n \Lambda^{(0)} \ , \qquad (\lambda_L < 1) \ . \tag{8}$$

In the absence of the island's hierarchy, we get just the result (1) with $\Lambda = \Lambda^{(0)}$. In the presence of the island's hierarchy we can introduce particle flux $\Phi^{(n)}$ in phase space through the island's chain of the n-th hierarchical level. It reads

$$\begin{aligned} \Phi^{(n)} = S^{(n)} N^{(n)} / T^{(n)} &= \Phi^{(0)} (\lambda_S \lambda_N / \lambda_T)^n \ , \\ \Phi^{(0)} &= S^{(0)} N^{(0)} / T^{(0)} \end{aligned} \tag{9}$$

in correspondence to (7) and (8). For $K = K^*$ and the corresponding island's hierarchy $\lambda_N = \lambda_T$ [9] and thus

$$\Phi^{(n)} = \lambda_S^n \Phi^{(0)} \ . \tag{10}$$

The quantum mechanical consideration of the proliferation of islands is meaningful until the smallest island size is of the order of $\hbar$. Therefore we get from (7),(9),(10):

$$\begin{aligned} S_{\min} = \hbar = S^{(n_0)} &= S^{(0)} \lambda_S^{n_0} \ , \\ \Phi_{\min} = \Phi^{(0)} \lambda_S^{n_0} = \lambda_S^{n_0} S^{(0)} N^{(0)} / T^{(0)} &= \hbar N^{(0)} / T^{(0)} \ , \end{aligned} \tag{11}$$

and the quantum "cut-off" of the hierarchy appears at

$$n_0 = |\ln(\hbar / S^{(0)})| / |\ln \lambda_S| \equiv |\ln \tilde{h}| / |\ln \lambda_S| \ , \tag{12}$$

where we introduce a dimensionless semi-classical parameter $\tilde{h} = \hbar / I_0$ and specify $S^{(0)} \equiv I_0$. Particularly, for the hierarchy at $K = K^*$ we have $N^{(0)} = 3$ and $\lambda_N = 8$ but it could be many other hierarchies (see more in [24]). After the substitution $\Lambda = \Lambda^{(n_0)}$ we get with (8) and (12):

$$\tau_\hbar = (1/\Lambda^{(n_0)}) \ln(1/\tilde{h}) = (1/\tilde{h})^{1/\mu} \ln(1/\tilde{h}) \tag{13}$$

with

$$\mu = |\ln \lambda_S| / \ln \lambda_T \ . \tag{14}$$

The expression of μ through λ_S, λ_T was found in [23] for the considered islands hierarchy and the expression for $\tau_\hbar$ coincides with the obtained in [17] in a different way. The expression (13) is close to (3) up to a logarithmic term and defines

$$\delta = 1/\mu, \tag{15}$$

which for $K = K^*$ provides $\mu = 1.25$ (see [9]) and $\delta = 0.8$. When there is no islands hierarchy, we may put $\lambda_S \to 0$ or $n_0 \to 0$. This yields transition from the algebraic law (13) to the logarithmic one for the breaking time and resolves the paradox discussed in [4]. In addition to this, it was shown in [9] that for the considered islands hierarchy

$$\gamma = 1 + \mu = 1 + |\ln \lambda_S|/\ln \lambda_T \ , \tag{16}$$

what gives $\gamma = 2.25$ for $K = K^*$. Just these values of γ and δ we have checked by a simulation.

3. Numerical study of the problem is based on investigation of the quantum survival probability in some domain $\Delta p \in (-\pi, \pi)$ that includes the islands hierarchy [25]. The main difficulty appears due to a part of the wave function that belongs to an island interior and "flies" fast along p. A simple way to avoid this type of the "escape" from Δp is to apply a shift operator

$$\hat{J} = \exp\{-(2\pi/\tilde{h})\partial/\partial\hat{n}\} = \exp(-2\pi i q/\tilde{h}) \ , \tag{17}$$

where q is a dimensionless coordinate, $\hat{p} = \tilde{h}\hat{n} = -i\tilde{h}\partial/\partial q$ is a dimensionless momentum operator and the wave function Ψ_t at a discrete time t is considered in the coordinate space. In analogy to [11,26] we also introduce the absorbing boundary conditions at the edges of the interval $\Delta n \in [-(N+1)/2, (N-1)/2]$ for the momentum eigenvalues $\tilde{h}n$. Then the quantum map that keeps the information of the trapping into Δn part of the wave function is

$$\Psi_{t+1} = \hat{\mathcal{P}}\hat{J}\hat{U}\Psi_t \tag{18}$$

with the evolution operator

$$U = \exp\big(-i\tilde{h}\hat{n}^2/2\big)\exp\big(-i(K/\tilde{h})\cos q\big) \ . \tag{19}$$

As usual, for the numerical convenience the dimensionless Planck constant $\tilde{h}$ is taken in a form $\tilde{h} = 2\pi/(N+g)$, where $g = (\sqrt{5}-1)/2$ is the inverse golden mean.

The survival probability is defined as

$$P(t) = |\Psi_t|^2 \tag{20}$$

together with definition (18). Simulation was performed for 12 values of N from the interval $(5 \cdot 10^3 \div 7.5 \cdot 10^4)$ which corresponds to a good semi-classical approximation for a fairly long time until it fails.

A typical behavior of $P(t)$, obtained for $K = K^*$, is shown in Fig. 1. It consists of the crossover from the classical behavior (2) with $\gamma \approx 2.25$ (the same as in [9]) to some very different dependence. The crossover points $\tau_\hbar$ can be identified in a way as it is shown in Fig. 1 for different values of N, i.e. $\tilde{h}$. The corresponding result is presented in Fig. 2. It provides the value of $1/\delta \approx 1.33 \pm 0.36 \approx \mu$ in a good agreement to the presented theoretical estimation (see (15)).

4. In conclusion, we need to underline that there is no unique scenario of chaotic diffusion in classical limit and therefore one may expect no unique breaking mechanism of quantum classical correspondence. Stickiness and algebraic kinetics through cantori was considered in [8] with a specific choice of the Markov tree that defined a type of kinetics and corresponding scales. In this consideration a special hierarchical set of resonance islands was selected, while in [9,23] the Markov tree was constructed for the tangle islands (see more about the classification in [20]). The difference in the choice of the islands set selection is imposed by the value of K. In this article we choose $K = K^*$ which leads to (15) while in [16,27,28] the value $K \le 2\pi$ was selected which probably leads to stickiness phenomenon described in [8] and to the value

$$\delta = 1/\gamma \ . \tag{21}$$

This value was not linked to the transport exponents μ. Let us mention also the value $\delta \sim 0.5$ proposed in [26] for $K = 2.5$ when sticky islands set appears without accelerator mode and with no superdiffusion.

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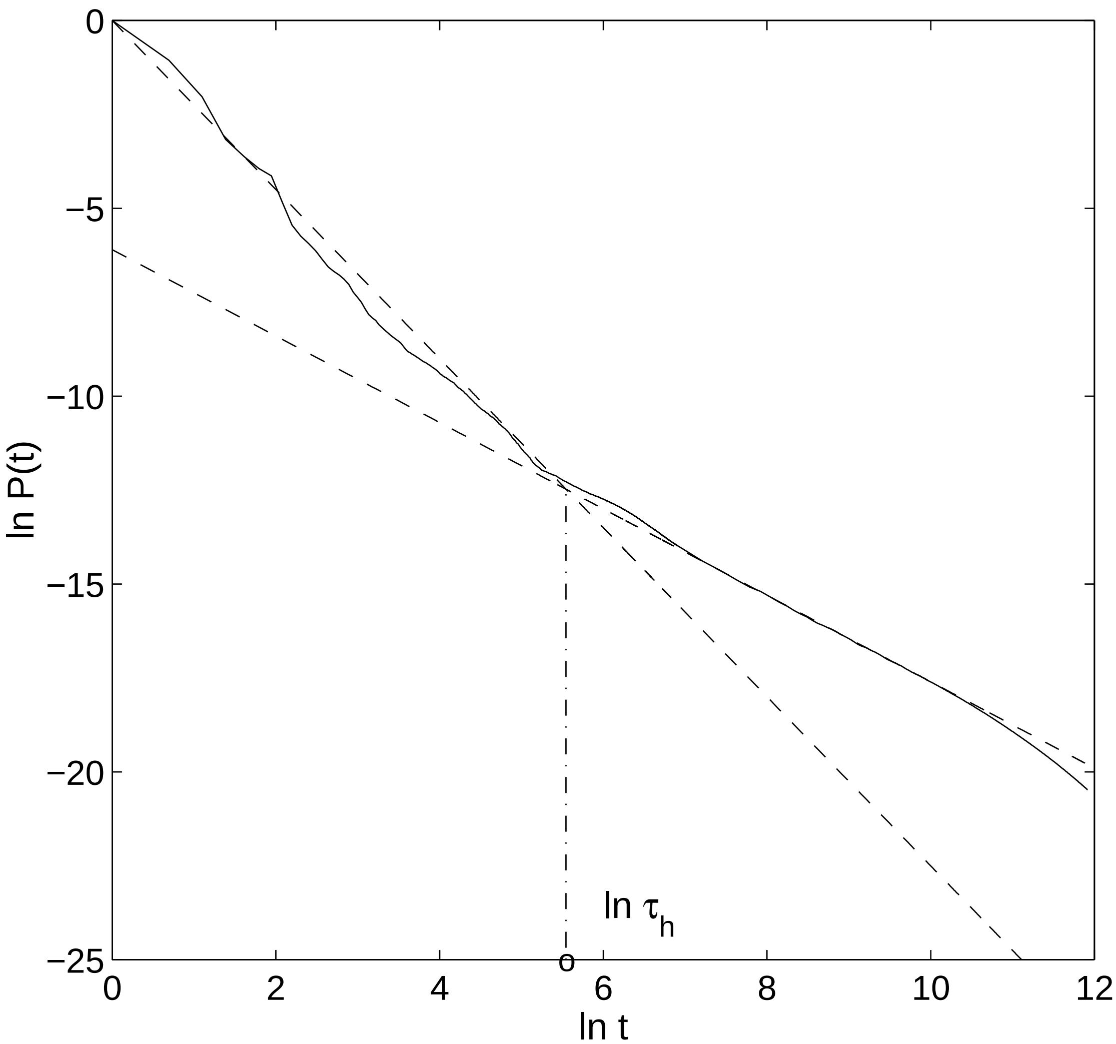
FIGURE CAPTIONS

Fig. 1 Typical evolution of the quantum survival probability for $N = 25557$. The dashed lines, which correspond to the asymptotics slopes, determine the breaking point $\tau_\hbar$.

Fig. 2 The quantum breaking points $\tau_\hbar$ vs dimensionless semiclassical parameter $\tilde{h}$. The solid line with the slop $1/\delta \approx 1.33 \pm 0.36$ corresponds to least square calculations, and the dashed line is the analytical prediction with $\delta = 1/\mu$.



0
−5
−10
−15
−20
−25
ln P(t)
0
2
4
6
8
10
12
ln t
ln τ_h
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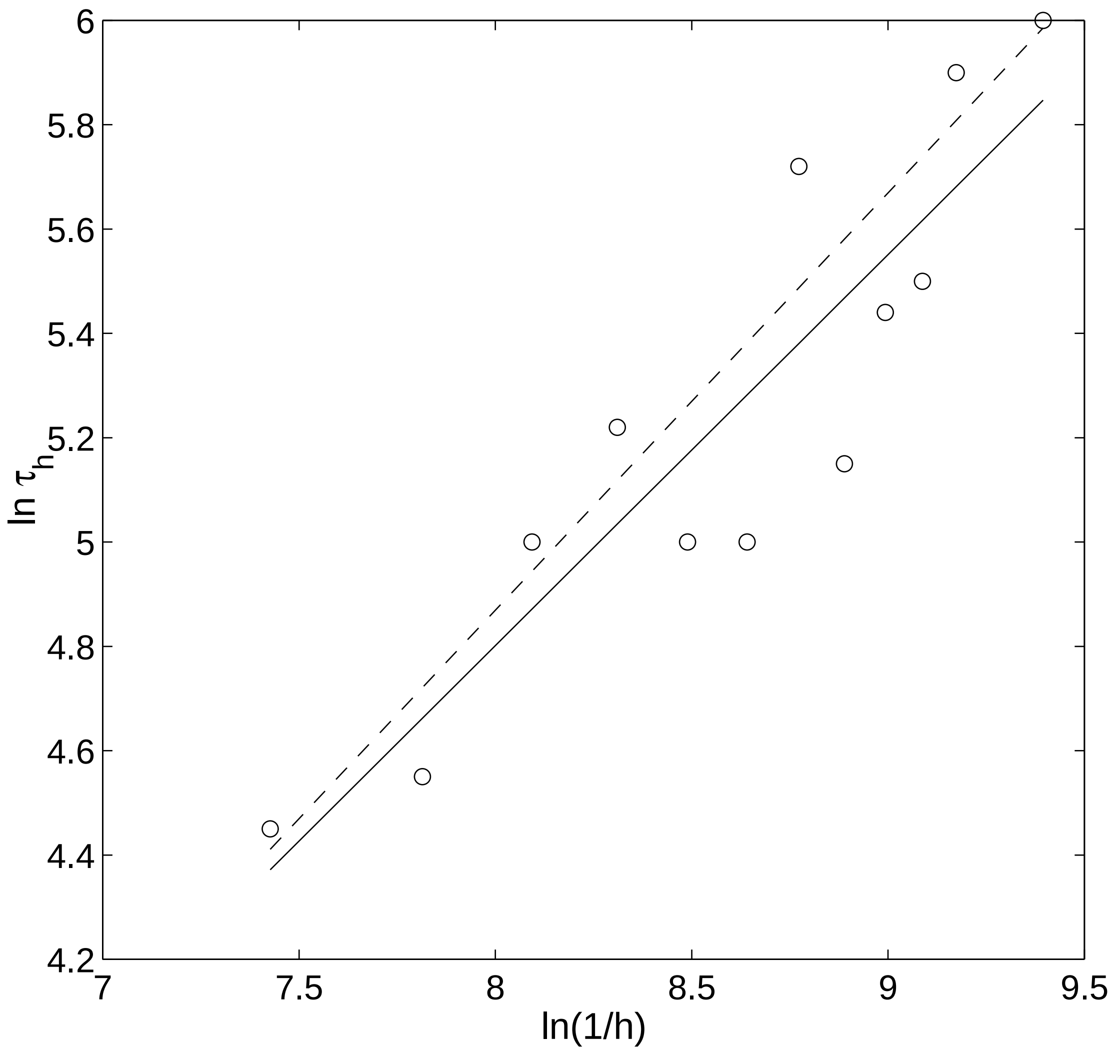

6
5.8
5.6
5.4
5.2
5
4.8
4.6
4.4
4.2
7
7.5
8
8.5
9
9.5
ln(1/h)
ln τ_h